A SET OF POSTULATES FOR THE LOGICAL STRUCTURE OF MUSIC

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EVERY universe of discourse has its logical structure. There is a certain number of possible situations that may occur in it. Expert chess-players have a fair, though probably not explicit, knowledge of the situations that may be created by manipulating the thirty-two pieces over sixty-four squares; and the mediaeval logicians have expressed all the possible contortions of the syllogism in that curious rune: "Barbara, celarent, Darii," *etc.* Such a collection of hypothetical situations comprises an empirical study of the "field" (to borrow a term from physics), within which any specific case may occur.

In a very great or complex universe of discourse, however, a simply enumerative inventory is not practicable. The possible configurations of the chess-board, for example, run into such staggering numbers that an encyclopaedic knowledge of them would be useless because of its vastness, even if any mind could retain it; it could not be surveyed at a glance. And chess is a system of fair simplicity and obviousness. Our systems of science, morality, art, and all the mazes of practical life, are so enormously complex that frequently they are not even viewed as definite logical fields. We cannot hope to exhaust their possibilities by a perfect induction; our only hope is, to find certain formal relations obtaining among their elements, which will serve as a key to the whole storehouse of possibilities. This is the method of mathematics, and of physics by virtue of its mathematical form.

Modern logic, being far more complex than the Aristotelian truisms, has resorted to a search for very general properties, resulting in the formulation of the so-called "Boolean" algebras. Such an algebra contains, in a few postulates of great generality, all the possible vicissitudes within the universe of propositions. Furthermore, it has the interesting property of being applicable, with few if any modifications, to other than propositional systems. This amenability of the Boolean structures to various *interpretations* has called attention to the fact that every system has general properties, and could theoretically be reduced to some postulate-set. In most cases, of course, this possibility remains in the umbral regions of theory, because of the great complexity of material to be analyzed; Leibniz remarked that a perfect mathematician could find the equation to the curve of any familiar profile, but none of us would undertake to compute profile-curves (to be registered, perhaps, like finger-prints). Nevertheless, there are certain systems, other than those of Boolean character or of ordinary mathematics, which are simple and obvious enough to let their formal properties be described.

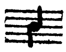
A good case in point is the structure of music. It seems plausible that there are fairly few sorts of elements involved in music, and that there are just certain possibilities of combining these according to definite principles. A set of such principles to delimit the field in which any musical configuration whatever must necessarily lie, constitutes the abstract form, the logic of music, and is itself of the nature of a special algebra, neither "numerical" nor "Boolean," but of equally mathematical form, amenable to at least one interpretation. The following postulate-set is designed to embody this abstract structure:

We assume a class, K, of elements $a, b, c, \ldots$. a binary operation $\cdot$, a binary operation $\rightarrow$, a monadic relation (or property) C, and a dyadic relation $<$.

1. If a and b are any K-elements, then $a \cdot b$ is a K-element.
2. For any K-element a, $a \cdot a = a$.
3. If a and b are any K-elements, $a \rightarrow b$ is a K-element.
4. For any K-elements a and b, $a \rightarrow b = b \rightarrow a$ implies $a = b$.
5. For any K-elements a, b and c, $(a \cdot b) \cdot c = b \cdot (a \cdot c)$.
6. For any K-elements a, b and c, there is at least one K-element d, such that $(a \rightarrow b) \cdot (c \rightarrow d) = (a \cdot c) \rightarrow (b \cdot d)$.
7. There is at least one K-element r, such that for any K-element a, $a \cdot r = a$.
8. There is a K-subclass T, such that for any K-elements a and b, other than r, and any K-element c, if $a = b \cdot c$ implies $b = c$, and $a = b \rightarrow c$ implies $b = r$ or $c = r$, then a is a T-element.
9. For any T-element a, $C(a \cdot a)$ holds.
10. For any K-elements a, b, and c, $\backsim C(a \cdot b)$ implies $\backsim C(a \cdot b \cdot c)$.
11. For any K-element a there is a K-subclass A such that for any K-elements b and c, b is an A-element always and only if $Ca \cdot c \equiv Cb \cdot c$.
12. For any distinct T-elements a and b, $\backsim (a < b) \equiv (b < a)$.
13. For any T-elements a, b and c, $a < b$ and $b < c$ implies $a < c$.
14. For any T-element a, and any other T-element b which is not an A-element, and for any A-element a', there is at least one B-element, b', such that if $a < a'$, then $\backsim (a < b < a')$ implies $(a < b' < a')$.
15. For any T-element a there is at least one A-element a° such that for any A-element b, distinct from both a and a°, $a < b$ and $a < a^\circ$ implies $a^\circ < b$, and $a < a^\circ$ and $b < a^\circ$ implies $b < a$.

This postulate set, viewed purely as a structure without any hint of a possible interpretation, looks enough like Boolean algebra to make the differences between the two contrastable.

Its two operations, $\cdot$ and $\rightarrow$, are suggestive of Boolean multiplication and addition. Indeed, the operation $\cdot$ has all the characteristics of multiplication, or conjunction; it is commutative, associative, absorptive, etc., like the Boolean $\times$, which is sometimes expressed by the same symbol, $\cdot$. (I have retained the classic notation wherever the analogy is exact.) But the operation which might be thought to correspond to disjunction, has not the properties of logical $+$; for note, it is not commutative, and not absorptive. Here is our first important divergence from the system of Boole. Furthermore, although the unique element r corresponds in some respects (though not in all) to the Boolean 0, there is no analogue to 1. The relation $<$ has properties very similar to those of inclusion ("implication"), although it is strictly a serial relation, like that of magnitude; but this limitation, as well as the fact that it applies only within a sub-class of our initial K, might be regarded as merely a special restriction on an algebra otherwise derivable from the classic premises. The presence of the monadic relation or *property* C is suggestive of the use which Boole's successors made of his 1, expressed in the peculiarly muddled formula, "$a=(a=1)$." A corrected statement of the propositional calculus would replace this abuse of 1 by some value-function such as Ca, to be read: "a is true." So the fundamental differences between the two systems are due, in the main, to (1) the non-commutativity of $\rightarrow$, and (2) the incomplete nature of the element correspondent to 0, and absence of any *element* 1. These two chief divergences make the new algebra less symmetrical than the logical calculus. The duality of $+$ and $\times$ is not preserved, and furthermore we lose the tidiness and symmetry due to the comprehension of the whole Boolean system between two definite limiting terms, 0 and 1. Whether the loss of simplicity is counterbalanced by any gain in scope or interest, is as yet a matter of futile conjecture.

Our interpretation of the new algebra can be stated in terms of the formal structure of music. It is not easy for people accustomed to the concepts usually employed in musical theory to deal with such abstract forms as, e. g., "musical element," including all tones, intervals, progressions and rests; to conceive a counterpoint as one interval of several progressions, or one progression of several intervals; or to treat  or  as intervals (although persons with a certain degree of sophistication in music are content to accept  as a perfectly respectable interval). In analyzing the basic relations of all possible musical structures we must forget such European conceptions as the diatonic scale, and above all we must be free from that popular tyrant, the piano, with its peculiar gradation of pitch by "half-tones." The determination of specific intervals, major, minor, perfect and altered, are special postulates, and do not concern the truly fundamental requirements of music as such; our postulate-set might be termed a prolegomenon to any music. Our present system embodies neither the laws of composition (though these may be specified within it) nor the physics of tone (which is another story).

By interpretation, then, our postulates may be read as follows:

1. If a and b are any musical elements, the interval a-with-b is a musical element.

2. If a is any musical element, a is equal to the unison a-with-a.

3. If a and b are any musical elements, the progression a-to-b is a musical element.

4. If a and b are any musical elements, and if the progression a-to-b = b-to-a, then a and b are the same musical element.

5. If a, b and c are any musical elements, then the

interval (a-with-b)-with-c is the same as the interval b-with-(a-with-c).

6. If a, b and c are any musical elements, then there is at least one musical element d, such that the interval of progressions, (a-to-b)-with-(c-to-d), is equal to the progression of intervals, (a-with-c)-to-(b-with-d).*

7. There is at least one musical element r, such that, if a is any other musical element, the interval a-with-r is equal to a.

8. There is a subclass, T, of musical elements, namely Tones, such that if a and b are any musical elements distinct from r, and c is any musical element, and if ($a =$ b-with-c) implies ($b = c$), and ($a = b$-to-c) implies either $b = r$ *or* $c = r$, then a is a tone. (I. e., if a is an interval it is a unison, and if a is a progression every member but one is a rest).

9. If a is any tone, the unison a-with-a is consonant.

10. If a, b, and c are any musical elements, then if a-with-b-with-c is consonant, a-with-b is consonant.

11. For any musical element a, other than r, there is a subclass of elements, A, ("recurrences" of a,) such that for any elements b and c, b is a recurrence of a if and only if "a-with-c is consonant" is equivalent to "b-with-c is consonant."

12. If a and b are any distinct tones, then if a is not before b in order of pitch, b is before a in order of pitch.

13. If a, b and c are any tones, then if a is before b and b is before c in order of pitch, a is before c.

14. If a is any tone, and b is any tone distinct from a and not a recurrence of a, and a' is any recurrence of a, then there is at least one b', a recurrence of b, such that if a is before a', and b is not between a and a', then b' is between a and a' in order of pitch.

* This postulate embodies the principle of counterpoint.

15. For any tone a, there is a tone a°, a recurrence of a, such that if b is any other recurrence of a, and if a is before b in order of pitch, and also a is before a°, then a° is before b; and if a is before a° and b is before a°, then b is before a in order of pitch (i. e., there is at least one recurrence of a, the *octave*, such that no other recurrence can lie between it and a).

From this postulate-set we may deduce all the essential relations among musical elements, such as the repetitional character of the order of tones within the octave, the equivalence of consonance-values of any interval and any repetition of itself, the recurrence of an interval of given relative pitch in succeeding octaves, etc. We adduce some of the most important theorems below:

Theorem 1.

For any K-elements a, b: $a\cdot b=b\cdot a$

Proof:

	$b=b\cdot r$	(by 7)
therefore	$a\cdot b=a\cdot(b\cdot r)$	
	$=(b\cdot a)\cdot r$	(by 5)
	$=b\cdot a$	(by 7)
therefore	$a\cdot b=b\cdot a$	Q.E.D.

Theorem 2.

For any K-elements a, b, c: $(a\cdot b)\cdot c=a\cdot(b\cdot c)$

Proof:

	$(a\cdot b)\cdot c=b\cdot(a\cdot c)$	(by 5)
	$=(a\cdot c)\cdot b$	(by theorem 1)
	$=(c\cdot a)\cdot b$	(by theorem 1)
	$=a\cdot(c\cdot b)$	(by 5)
	$=a\cdot(b\cdot c)$	(by theorem 1)
therefore	$(a\cdot b)\cdot c=a\cdot(b\cdot c)$	Q.E.D.

Theorem 3.

For any T-elements a, b and any A-element a' and any B-element b': $Ca\cdot b\equiv Ca'\cdot b'$

Proof:

	$Ca \cdot b \equiv Ca' \cdot b$	(by 11)
	$a' \cdot b = b \cdot a'$	(by theorem 1)
	$Cb \cdot a' \equiv Cb' \cdot a'$	(by 11)
	$b' \cdot a' = a' \cdot b'$	(by theorem 1)
therefore	$Ca \cdot b \equiv Ca' \cdot b'$	Q.E.D.

Theorem 4.

For any T-element a and any A-element a': $Ca \cdot a'$

Proof:

	$Ca \cdot a$	(by 9)
	$Ca \cdot a \equiv Ca' \cdot a$	(by 11)
	$a' \cdot a = a \cdot a'$	(by theorem 1)
therefore	$Ca \cdot a'$	Q.E.D.

Theorem 5.*

*For the sake of brevity in the following proofs I shall employ the accepted symbols ϵ, $\supset$, $\exists$, respectively for "is a member of," "implies," and "there exists."

For any T-elements a, b: $b \epsilon A \cdot \supset \cdot a \epsilon B$

Proof:

For any K-element c: $b \epsilon A \cdot \supset \cdot Cb \cdot c \equiv Ca \cdot c$	(by 11)
$Cb \cdot c \equiv Ca \cdot c \cdot \supset \cdot a \epsilon B$	(by 11)
therefore $b \epsilon A \cdot \supset \cdot a \epsilon B$	Q.E.D.

Theorem 6.

For any T-elements a, a°, b and for any B-element b': $(a < b < a^\circ).(a < b').\supset .a^\circ < b'$

Proof:

	$a < b \,.\supset .\, \sim (b < a)$	(by 12)
assume	$(a < b < a^\circ) \,.\, (a < b' < a^\circ)$	
then either	$a < b < b' < a^\circ$ or $a < b' < b < a^\circ$	
but	$(\exists a')$: $\sim (b < a < b') \,.\supset .\, b < a' < b')$	(by 14)
and	$b' \epsilon B \,.\supset .\, b \epsilon B'$	(by theorem 5)
then	$\sim (b' < a < b) \,.\supset .\, b' < a' < b$	(by 14)
but	$(a < b < b' < a^\circ) \,.\, (b < a' < b') \,.\supset .\, a < a' < a^\circ$	
and	$(a < b' < b < a^\circ) \,.\, (b' < a' < b) \,.\supset .\, a < a' < a^\circ$	

contrary to 15.

Therefore $(a < b < a^\circ) . (a < b') : \supset . a^\circ < b'$ Q.E.D.

Theorem 7.

For any T-elements a, b, c, a°, b°: there is a C-element c° such that $(a < b < a^\circ < b^\circ) . (a < c < b) : \supset . (a^\circ < c^\circ < b^\circ)$

Proof:

$(a < c < b) . (b < a^\circ) : \supset . a < c < a^\circ$ (by 13)

and $(\exists c') : . (c < b) . (b < b^\circ) : \supset . b < c' < b^\circ$ (by 14)

but $\sim (a < c < c' < a^\circ)$ (by theorem 6)

therefore $a^\circ < c' < b^\circ$

Lemma:

For any C-element x, distinct from c and from c':

$$\left.\begin{array}{l} a < c < a^\circ . \supset . \sim (a < x < a^\circ) \\ b < c' < b^\circ . \supset . \sim (b < x < b^\circ) \end{array}\right\} \quad \text{(by theorem 6)}$$

$$\left.\begin{array}{ll} \text{then} & c < x . \supset . c' < x \\ \text{and} & x < c' . \supset . x < c \end{array}\right\} \quad \text{(by theorem 6 and Hyp)}$$

hence $c' = c^\circ$ (by 15)

therefore $(a < b < a^\circ < b^\circ) . (a < c < b) : \supset . a^\circ < c^\circ < b^\circ$ Q.E.D.

There are probably many other relations among musical elements, to be derived from this postulate-set; but even a complete development of it can give us only the *general,* or essential, musical possibilities. The particular structures employed in traditional European music require such further specifications as a next-member postulate for the series generated by $<$, determination of the consonant intervals other than unisons and repetitions, the introduction of T-functions ♯ and ♭, and possibly one or more other additional notions. By imposing alternative sets of restrictions upon our original K, we can derive different types of music. For instance, a postulate-set for Hawaiian music would contain a continuous-series postulate for pitch instead of a next-member postulate; in Gaelic music, adjacent tones of the scale do not yield dissonant intervals, as in diatonic; in ancient Greek harmony the major third was treated as a dissonance. In the future I hope to establish a set of special premises sufficient for the analysis of an orthodox figured bass, for with such material at our command we could feel confident that the complications and vagaries of modern music could by ingenious manipulation be expressed as well as the simpler, more standardized forms.

There is a further point of interest in this attempt to discern the purely logical structure of the musical universe —a matter of such philosophical import, howbeit of such unsubstantiated, visionary character, that I offer it as the merest suggestion: is it possible that music is not the only interpretation for this algebra, but that some logician versed in the arts, especially in arts other than music, might trace similar structures in some other form of aesthetic expression? The implication of such a hypothesis for the philosophy of art is obvious and vital. Psychology and metaphysics alike have failed so far to put aesthetics on any better basis than a purely empirical one; is it conceivable that logic might bridge the gap between those two disciplines and discover truly fundamental principles whereon to build a rational science of aesthetics? I have added this speculative paragraph with hesitation, with the discomfort which a mere logician quite properly feels in the presence of philosophical problems; but add it I must, even as a fantastic hypothesis, the timid, scientific version of Schopenhauer's bold poetic dictum, *"die Baukunst ist erstarrte Musik."*

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